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Richardson–Sielecki Schemes for the Shallow-Water Equations, with Applications to Kelvin Waves

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Using Kelvin waves for illustrative purposes, this paper shows that a common finite-difference scheme, used in tide and storm-surge models, actually distorts long waves by changing the orientation of wave crests and troughs. The magnitude of these errors is discussed for a simple model representing the North Sea. Modifications are discussed which correct these effects without sacrificing the computational economy of the original scheme.

1. INTRODUCTION

Finite-difference schemes based on a single Richardson grid [1] have the advantage that they minimize the number of variables which must be computed for a given degree of spatial resolution. Certain of these schemes achieve further computational economy by using only the most recently computed value of the variable appropriate to each grid point. Consequently, no memory space is needed for values at other time levels and no time is spent copying data arrays simply in order to update variables.

One such scheme has been used quite extensively for the shallow-water equations in model studies of tides and storm surfes [2–4] and can be conveniently designated in such applications as the Richardson–Sielecki scheme since it involves calculating variables on a Richardson grid using a particular method of handling the Coriolis terms introduced by Sielecki [5]. One of the purposes of this paper is to illustrate hitherto unsuspected distortion of long waves produced by this difference scheme. Later a modified Richardson–Sielecki scheme introduced by Prandle [6] is analysed and shown to eliminate the unwanted effects experienced with the original scheme.

Numerical tests were carried out for a particularly simple case, Kelvin waves in a uniform depth channel governed by the linearized, shallow-water equations. This permits comparison with analytic solutions to the relevant difference equations and permits quantitative study of the distortions in question, unobscured by initialization and truncation errors which are unavoidable in more complicated models. Similar effects must occur in numerical models of realistic long-wave problems and introduce errors which can substantially affect the validity of the results.

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2. Kelvin Waves

In the linearized form relevant here, the shallow water equations are

$$\eta_t = -h(u_x + v_y), \tag{1a}$$

$$u_t = -g\eta_x + fv, \tag{1b}$$

$$v_t = -g\eta_y - fu, \tag{1c}$$

where η is surface elevation relative to mean water level, u and v are velocity components in a horizontal Cartesian coordinate system (x, y), f is the Coriolis parameter and h is the uniform mean water depth. Essentially these equations are adequate for explaining the behaviour of all long waves such as tsunamis, surges, seiches and tides in uniform depth basins where advection and friction are negligible.

The most general wavelike solution to Eqs. (1) can be written as

$$\eta = A_\eta \exp[i(\omega t + kx + ly)], \tag{2a}$$

$$u = A_u \exp[i(\omega t + kx + ly)], \tag{2b}$$

$$v = A_v \exp[i(\omega t + kx + ly)], \tag{2c}$$

where A_η, A_u, A_v are interrelated complex amplitudes; ω is frequency (assumed positive); k and l are wave numbers in the x and y directions, respectively; and c is the wave speed, given by $c = (gh)^{1/2}$. Substituting (2) into (1) leads eventually to the following dispersion relation for long waves in uniform depth water:

$$\omega^2 = c^2(k^2 + l^2) + f^2. \tag{3}$$

Central to the following discussion is a special class of solutions, known as Kelvin waves, which can propagate parallel to a straight coastline, with zero transverse velocity. They are prototypes of most tides in real ocean basins. Without loss of generality, one can narrow discussion to the case of a coast coincident with the x-axis. If the water-filled basin occupies the positive y half-plane, the only physically realizable Kelvin wave has

$$k = -\frac{\omega}{c}; \qquad l = i\frac{f}{c}, \tag{4}$$

the corresponding solution being

$$\begin{aligned} \eta &= Be^{-fy/c}\sin(\omega t - |k|\,x + \phi), \\ u &= \frac{gB}{c}e^{-fy/c}\sin(\omega t - |k|\,x + \phi), \\ v &= 0, \end{aligned} \tag{5}$$

where B is an arbitrary real amplitude and ϕ is an arbitrary phase angle. The notations $-|k|$ and $+|k|$ will be used wherever necessary to indicate that a wavenumber is negative (progressive wave) or positive (regressive wave). Where a wavenumber may assume positive or negative values, it is written simply as k. When the basin occupies the negative y half-plane, it is possible to have

$$k = \frac{\omega}{c}; \qquad l = -i\frac{f}{c} \tag{6}$$

with corresponding solution

$$\begin{aligned} \eta &= Be^{fy/c}\sin(\omega t + kx + \phi), \\ u &= -\frac{gB}{c}\,e^{fy/c}\sin(\omega t + kx + \phi), \\ v &= 0. \end{aligned} \tag{7}$$

It can be seen that in both instances the Kelvin wave must have the coast on its right (in the northern hemisphere) relative to the direction of travel and that wave crests and troughs are perpendicular to the coast. Wave amplitude decreases exponentially in the seaward direction.

In a straight channel, Kelvin waves can propagate along the channel in both directions. For instance, a channel with sides parallel to the x-axis (west–east axis) may have progressive Kelvin waves of type (5) bound to the southern shore and regressive waves of type (7) bound to the northern shore.

3. The Richardson Grid

Adoption of a Richardson grid as the basis for a finite-difference approach to solving the shallow-water equations (1) implies that the dependent variables η, u and v are to be determined at points positioned relative to one another in (x, y, t)-space as shown in Fig. 1. Many of the other grids used for discrete shallow-water equations are superpositions of two or more Richardson grids [7]. Provided that a single Richardson grid provides appropriate points with which to implement some specified difference scheme, its use will minimize the number of values of η, u and v which have to be calculated in order to achieve a given spatial resolution over the model domain.

Some system of indexing the variables on the grid is necessary and here the notation η^s_{mn}, $u^{s+1/2}_{mn}$, $v^{s+1/2}_{mn}$, will be used to indicate the variables whose spatial layout is shown in Fig. 2. The superscripts refer to the relevant time levels indicated in Fig. 1.

For economy in computing finite differences, the grid interval sizes Δx and Δy are usually held constant throughout the model domain. The choice of values for Δx and Δy in a given model is normally based on the spatial resolution required.

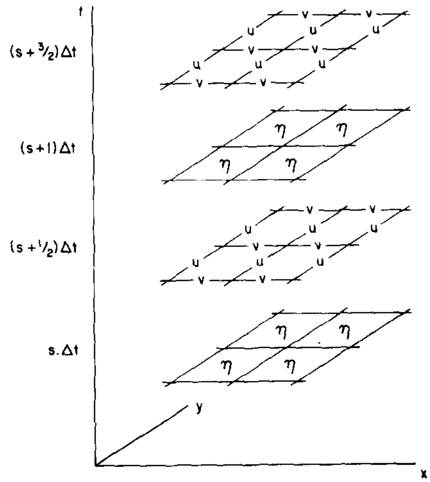

FIG. 1. Portion of a Richardson grid.

4. THE RICHARDSON–SIELECKI FINITE-DIFFERENCE SCHEME FOR THE SHALLOW-WATER EQUATIONS

To understand the origins of the Richardson–Sielecki (henceforth R–S) scheme for the shallow-water equations it is useful to look at two schemes used for the simple wave equations in two dimensions:

$$\begin{aligned} \eta_t &= -h(u_x + v_y), \\ u_t &= -g\eta_x, \\ v_t &= -g\eta_y. \end{aligned} \tag{8}$$

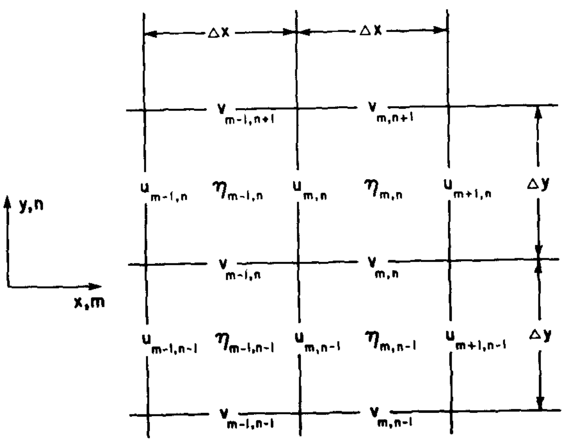

FIG. 2. Notation of discretized variables.

Using the notations of Section 3 and a Richardson grid, consider first the simple time- and space-centered scheme

$$\begin{aligned}
\frac{\eta_{mn}^{s+1}-\eta_{mn}^{s}}{\Delta t} &= -h\left[\frac{u_{m+1,n}^{s+1/2}-u_{mn}^{s+1/2}}{\Delta x}+\frac{v_{m,n+1}^{s+1/2}-v_{mn}^{s+1/2}}{\Delta y}\right],\\
\frac{u_{mn}^{s+3/2}-u_{mn}^{s+1/2}}{\Delta t} &= -g\,\frac{\eta_{mn}^{s+1}-\eta_{m-1,n}^{s+1}}{\Delta x},\\
\frac{v_{mn}^{s+3/2}-v_{mn}^{s+1/2}}{\Delta t} &= -g\,\frac{\eta_{mn}^{s+1}-\eta_{m,n-1}^{s+1}}{\Delta y}.
\end{aligned} \tag{9}$$

If u_{mn}^{s}, v_{mn}^{s}, u_{mn}^{s+1}, v_{mn}^{s+1} are written in place of $u_{mn}^{s+1/2}$, $v_{mn}^{s+1/2}$, $u_{mn}^{s+3/2}$, $v_{mn}^{s+3/2}$ in (9), the so-called "forward-backward scheme" [7] is obtained. The notation of this second scheme implies that u- and v-points occur at the same time levels as the η-points; that is, the underlying grid differs from a Richardson grid (Fig. 1). But the schemes are essentially identical. Except in problems where both elevation and velocity are specified as functions of time at some boundaries, the numerical values of η, u, v computed with both schemes are identical, given the same initial state. The fact that scheme (9) in conjunction with a Richardson grid is the most appropriate interpretation of the computed values can best be illustrated by a simple example.

Taking the case of a plane sinusoidal wave propagating in the positive x-direction, the exact solution to difference equations (9) is then

$$\eta_{mn}^{s} = B\sin[\omega s\cdot\Delta t-|k|\,m\cdot\Delta x+\phi], \tag{10a}$$

$$u_{mn}^{s+1/2} = (gB/c)\sin[\omega(s+\tfrac{1}{2})\Delta t-|k|(m-\tfrac{1}{2})\Delta x+\phi], \tag{10b}$$

$$v_{mn}^{s+1/2} = 0, \tag{10c}$$

where Δt is the time step between two successive evaluations of a given variable. The presence of the factors $(s+\frac{1}{2})\Delta t$ and $s\cdot\Delta t$ in (10b) and (10a) indicate that in effect the velocity u_{mn} defined in (10b) pertains to a time level which is $\frac{1}{2}\Delta t$ later than that of η_{mn} defined in (10a); that is, the relative position in time of these variables is as shown in the Richardson grid in Fig. 1.

If, on the other hand, the notation $u_{mn}^{s}, v_{mn}^{s},\ldots$ is used in place of $u_{mn}^{s+1/2}$, $v_{mn}^{s+1/2},\ldots$ giving the forward–backward scheme, with u_{mn}^{s}, v_{mn}^{s} apparently on the same time level as η_{mn}^{s}, it is natural to take the progressive plane wave in question to be

$$\eta_{mn}^{s} = B\sin[\omega s\cdot\Delta t-|k|\,m\cdot\Delta x+\phi], \tag{11a}$$

$$u_{mn}^{s} = (gB/c)\sin[\omega s\cdot\Delta t-|k|\,(m-\tfrac{1}{2})\Delta x+\phi], \tag{11b}$$

$$v_{mn}^{s} = 0. \tag{11c}$$

(The factor $(m-\frac{1}{2})\cdot\Delta x$ in (11b) remains, expressing as it does the half-grid interval spacing between η_{mn}^{s} and u_{mn}^{s}.) However, if η, u and v are evaluated from Eqs. (11), say at $s = 0$, and used as initial conditions in the forward–backward scheme, it is found that in addition to the required progressive wave, a smaller regressive error wave is also generated. Whereas it has been common practice to use the notation of the forward–backward scheme and its implied grid for the shallow-water equations [2–4], in the following treatment a Richardson grid is presumed and the difference schemes involved are viewed as derived from (9), in order to eliminate the risk of errors of the type just described.

Turning to the linearized shallow-water equations (1), it can be seen that inclusion of the Coriolis terms causes some difficulty in any grid where velocities u and v are not calculated at coincident points. v has to be averaged in some manner for use in the equation for u, and vice versa. Further, in order to avoid numerical instability of the whole scheme, it is necessary to pay attention to the time levels of the velocity values used for the Coriolis terms. Sielecki [5] devised a stable scheme in which the velocities used for the Coriolis terms in the u and v equations come from different time levels. With the use of spatial averaging where necessary, on a single Richardson grid Sielecki's approach gives the following finite-difference equivalent to Eqs. (1):

$$\frac{\eta_{mn}^{s+1} - \eta_{mn}^{s}}{\Delta t} = -h\left[\frac{u_{m+1,n}^{s+1/2} - u_{mn}^{s+1/2}}{\Delta x} + \frac{v_{m,n+1}^{s+1/2} - v_{mn}^{s+1/2}}{\Delta y}\right], \tag{12a}$$

$$\frac{u_{mn}^{s+3/2} - u_{mn}^{s+1/2}}{\Delta t} = -g\frac{\eta_{mn}^{s+1} - \eta_{m-1,n}^{s+1}}{\Delta x} + \frac{f}{4}\left[v_{m-1,n}^{s+1/2} + v_{m-1,n+1}^{s+1/2} + v_{mn}^{s+1/2} + v_{m,n+1}^{s+1/2}\right], \tag{12b}$$

$$\frac{v_{mn}^{s+3/2} - v_{mn}^{s+1/2}}{\Delta t} = -g\frac{\eta_{mn}^{s+1} - \eta_{m,n-1}^{s+1}}{\Delta y} - \frac{f}{4}\left[u_{mn}^{s+3/2} + u_{m,n-1}^{s+3/2} + u_{m+1,n}^{s+3/2} + u_{m+1,n-1}^{s+3/2}\right]. \tag{12c}$$

Provided that all the η_{mn}^{s+1} are evaluated first, and then all the $u_{mn}^{s+3/2}$, this scheme has the economical property that only the most recently calculated value of each variable need be stored. For instance, the values $v_{mn}^{s+1/2}$, $v_{m,n+1}^{s+1/2}$,... used in (12b) are available from the previous time step when $u_{mn}^{s+3/2}$ is calculated and the updated values $u_{mn}^{s+3/2}$, $u_{m+1,n}^{s+3/2}$,... are all calculated just prior to their use in (12c). This permits use of single arrays of u and v values, $u_{mn}^{s+1/2}$ being overwritten by $u_{mn}^{s+3/2}$ as soon as the latter is calculated and similarly with v. As well as reducing storage requirements, this implies that no labelling of variables with respect to time level is required at the programming stage.

5. Stability and Dispersion Relations for the Richardson–Sielecki Scheme

If any solution to Eqs. (12) is unstable, i.e., grows exponentially with time, then the system is termed numerically unstable, since computational roundoff error can be counted on to initiate every unstable solution and the required solution is very soon swamped by growing error. For linear difference equations such as (12), a suitable stability criterion can be derived by considering the behaviour of the fundamental solutions, in terms of which all other solutions can be expressed.

A convenient form in which to write fundamental solutions to Eqs. (12) is

$$\begin{aligned} n^{s}_{mn} &= A_{n} \exp[i\{\omega s \cdot \Delta t + km \cdot \Delta x + ln \cdot \Delta y\}], \\ u^{s+1/2}_{mn} &= A_{u} \exp[i\{\omega(s+\tfrac{1}{2})\Delta t + km \cdot \Delta x + ln \cdot \Delta y\}], \\ v^{s+1/2}_{mn} &= A_{v} \exp[i\{\omega(s+\tfrac{1}{2})\Delta t + km \cdot \Delta x + ln \cdot \Delta y\}]. \end{aligned} \tag{13}$$

Here, A_{n}, A_u and A_v are constants determined by the initial conditions. Substituting (13) into (12) gives three simultaneous linear equations in A_{n}, A_u and A_v. For a non-trivial solution, the determinant of the corresponding coefficient matrix must equal zero. This leads to a cubic equation in $\lambda = \exp(i\omega \cdot \Delta t)$ which factorizes to give

$$\lambda - 1 = 0 \tag{14a}$$

and

$$\lambda^2 - (2 - A)\lambda + 1 = 0, \tag{14b}$$

where

$$A = 4c^2\Delta t^2 \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2}\right] + (f \cdot \Delta t \cos \tfrac{1}{2}k \cdot \Delta x \cos \tfrac{1}{2}l \cdot \Delta y)^2$$
$$- \frac{fc^2\Delta t^3}{\Delta x \cdot \Delta y} \sin k \cdot \Delta x \sin l \cdot \Delta y.$$

It is clear from the form of (13) that whether the solution is stable (bounded) or unstable (exponentially increasing) is determined by the magnitudes of the factors λ. For stability it is necessary that all three roots should satisfy

$$|\lambda| \leqslant 1. \tag{15}$$

The root defined by (14a) obviously satisfies (15); it corresponds to a steady state solution, $\omega = 0$. The two roots associated with (14b) are reciprocals of one another and hence the stability criterion (15) can be satisfied only if

$$|\lambda| = |\exp(i\omega \cdot \Delta t)| = 1, \qquad \text{i.e., } \omega \text{ is real.} \tag{16}$$

Rewriting (14b) with trigonometric instead of exponential forms leads to the following dispersion relation for waves in a Richardson–Sielecki model on a grid without boundaries:

$$\begin{aligned} \sin^2 \tfrac{1}{2}\omega \cdot \Delta t = {} & c^2\Delta t^2 \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2}\right] \\ & + (\tfrac{1}{2}f \cdot \Delta t \cos \tfrac{1}{2}k \cdot \Delta x \cos \tfrac{1}{2}l \cdot \Delta y)^2 \\ & - \frac{fc^2\Delta t^3}{4\Delta x \cdot \Delta y} \sin k \cdot \Delta x \sin l \cdot \Delta y. \end{aligned} \tag{17}$$

The stability condition (16) is equivalent to the requirement

$$|\sin \tfrac{1}{2}\omega \cdot \Delta t| \leqslant 1. \tag{18}$$

For feasible values of f, as Δt is increased the stability condition is violated first at the highest wavenumbers possible, which are [7]

$$k = \frac{\pi}{\Delta x}, \qquad l = \frac{\pi}{\Delta y}. \tag{19}$$

To avoid numerical instability, it is therefore necessary to limit the size of time step used to

$$\Delta t \leqslant c^{-1} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1/2}. \tag{20}$$

Analysis of stability is much more complicated when the model grid is of finite extent. The governing system of finite-difference equations has a finite number of fundamental solutions, whose wavenumbers depend on the geometry of the grid. Numerical experiments show, however, that when land boundaries are represented by a zero transport condition (e.g., $u \equiv 0$) or where the interaction with a neighbouring water body is represented by specifying the boundary velocity as a function of time, then the stability criterion (20) still applies.

6. Kelvin Wave Simulation with the Richardson–Sielecki Scheme

In order to understand the implications of the complicated dispersion relation (17) for the Richardson–Sielecki scheme, it is useful to introduce the idea of "relatively long" waves. A grid on which variables are sampled at points Δs apart can be used to model waves with wavenumbers up to $\pi/\Delta s$, since sampling theory dictates a minimum requirement of two samples per wavelength. However, to minimize errors due purely to the discretization, such as incorrect wave speed, it is always advisable to choose a grid spacing which gives at least 10 and preferably 20 or 30 sample points per wavelength for waves which are physically significant in the problem being modelled. At such high spatial resolution, where the waves in question are relatively long compared to the grid interval, the arguments of all the trigonometric terms in (17) are small and it is possible to write

$$\sin \tfrac{1}{2}\omega \cdot \Delta t \sim \tfrac{1}{2}\omega \cdot \Delta t, \qquad \sin \tfrac{1}{2}k \cdot \Delta x \sim \tfrac{1}{2}k \cdot \Delta x, \qquad \cos \tfrac{1}{2}k \cdot \Delta x \sim 1, \qquad \text{etc.}$$

Thus for relatively long waves, the dispersion relation (17) can be written approximately as

$$\omega^2 = c^2(k^2 + l^2) + f^2 - fc^2kl \cdot \Delta t. \tag{21}$$

Comparing this with (3), it can be seen that the method of representing the Coriolis terms in the R–S scheme introduces the spurious term $-fc^2kl\Delta t$ into the dispersion relation for relatively long waves. This causes at least two types of error. Discrete models are generally non-isotropic, in the sense that wave speed depends on direction of propagation. With some difference schemes (e.g., Fischer [8]), the non-isotropy is negligible for relatively long waves, but with the R–S scheme, the effects of the extra term in the dispersion relation do not decrease with wavenumber. For example, two relatively long waves with wavenumber vectors (k, l) and $(k, -l)$, which have the same wavelength $\lambda = 2\pi/(k^2 + l^2)^{1/2}$, have slightly different frequencies (and speeds) according to (21).

Another error introduced is the slewing of wavefronts, which can be illustrated explicitly by looking at the analogs of Kelvin waves in a Richardson–Sielecki model. Consider, as in Section 2, the case of waves bound to a coastline parallel to the x-axis. Assuming that the waves are relatively long, the relevant dispersion relation (17) can be satisfied through having

$$\omega^2 = c^2k^2$$

and

$$c^2l^2 - fc^2kl \cdot \Delta t + f^2 = 0.$$

There are two solutions of practical interest. Corresponding to Eq. (4) there is the case

$$k = -\omega/c; \qquad l = \beta + i\alpha, \tag{22}$$

where

$$\beta = \tfrac{1}{2}fk \cdot \Delta t, \qquad \alpha = (f/c)[1 - (\tfrac{1}{2}\omega \cdot \Delta t)^2]^{1/2}.$$

The corresponding solution, analogous to (5), has the form

$$\begin{aligned} \eta^s_{mn} &= Be^{-\alpha n \cdot \Delta y} \sin[\omega s \cdot \Delta t - |k|\, m \cdot \Delta x - \beta n \cdot \Delta y + \phi], \\ u^{s+1/2}_{mn} &= (gB/c)e^{-\alpha n \cdot \Delta y} \sin[\omega(s + \tfrac{1}{2})\,\Delta t - |k|\,(m - \tfrac{1}{2})\,\Delta x - \beta n \cdot \Delta y + \phi], \\ v^{s+1/2}_{mn} &= 0. \end{aligned} \tag{23}$$

The other case of interest is the regressive wave

$$k = \omega/c; \qquad l = \beta - i\alpha \tag{24}$$

(cf. Eq. (6)), which gives

$$\begin{aligned} \eta^s_{mn} &= Be^{\alpha n \cdot \Delta y} \sin[\omega s \cdot \Delta t + |k|\, m \cdot \Delta x + \beta n \cdot \Delta y + \phi], \\ u^{s+1/2}_{mn} &= -(gB/c)e^{\alpha n \cdot \Delta y} \sin[\omega(s + \tfrac{1}{2})\,\Delta t + |k|\,(m - \tfrac{1}{2})\,\Delta x + \beta n \cdot \Delta y + \phi], \\ v^{s+1/2}_{mn} &= 0. \end{aligned} \tag{25}$$

This form is the equivalent, in an R–S model, of solution (7) in the continuous case.

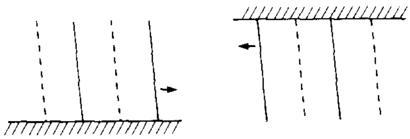

FIG. 3. Slewing of Kelvin waves by Richardson–Sielecki scheme. (——) Wave crest; (– – –) wave trough; (///) coastline; (→) direction of propagation.

Since $\omega \cdot \Delta t \ll 1$ for relatively long waves, $\alpha \simeq f/c$, which means that the rate of amplitude decay in the seaward direction differs only slightly between the R–S model and the continuous case. However, the fact that l has a non-zero real part, β, for the R–S scheme means that the phase increases linearly with distance from the coast; that is, crests (and troughs) are slewed backwards with respect to the direction of travel, as shown in Fig. 3, whereas in the continuous case, wave crests are perpendicular to the coast.

To gain some idea of the magnitude of the slewing effect, take as an example a rectangular basin closed at one end and driven by an incoming Kelvin wave at the open end. This type of model was used by Taylor to represent a semi-diurnal tide in the North Sea in his classical paper [9] on Kelvin and Poincaré waves. Suitable values for the model parameters are:

$$W = 544 \text{ km} \qquad \text{(basin width)},$$
$$f = 1.223 \times 10^{-4} \text{ sec}^{-1},$$
$$\omega = 1.4052 \times 10^{-4} \text{ sec}^{-1} \qquad (M_2 \text{ tide}),$$
$$g = 9.81 \text{ m/sec}^2,$$
$$h = 75 \text{ m}.$$

Assuming a Richardson grid with 16 grid intervals across the basin and with equal grid intervals in the x and y directions,

$$\Delta x = \Delta y = 34 \text{ km}.$$

With this grid, the maximum permissible time step, defined by (20), is $\Delta t = 886.3368$ sec; a value of $\Delta t = 885$ sec will be assumed in the subsequent calculations.

Using the above values, Eqs. (22) give

$$k = 5.1805 \times 10^{-6} \text{ m}^{-1},$$
$$l = \beta + i\alpha = 2.8036 \times 10^{-7} + i4.5001 \times 10^{-6} \text{ m}^{-1}.$$ [1]

[1] For comparison, it may be noted that the exact solution of Eq. (17), not using the "relatively long wave approximation, in this case is $k = 5.1839 \times 10^{-6}$ m^{-1}; $l = 2.8072 \times 10^{-7} + i4.4912 \times 10^{-6}$ m^{-1}.

Consequently, $\beta W = 0.153$ rad $= 8.74°$. This means that if the incoming semi-diurnal lunar tide in the North Sea is represented as a Kelvin wave bound to the English coast, southbound wavecrests are 12.7 min late on the opposite (European) coast in the R–S model. Errors of this nature, though possibly acceptable in some applications, could be troublesome in other cases, for example, if accurate location of amphidromic points is required.

The performance of R–S models is improved in some respects by removing the obvious asymmetry in the treatment of u and v and it is quite common practice to calculate the variables in the orders η, u, v and η, v, u on alternate steps. In other words, there is an exchange of advanced and retarded Coriolis terms on alternate time steps. However, the dispersion relation (21) is unaffected by this procedure, which means that the anistropy and slewing errors described above persist.

Using a finer grid does reduce these errors, since smaller values of Δx and Δy require use of smaller Δt to maintain computational stability and this in turn reduces the extraneous term in (21) which is the source of the errors. But this measure is costly in computer time and storage requirements and curing the problem by modifying the difference scheme, as described in the following section, is more practicable.

Although this discussion of errors in the R–S scheme has been limited to only relatively long, coastally bound waves, these play an important role in many tidal problems. Further, it is highly probable that similar undesirable effects occur with shorter waves for which the simplified dispersion relation (21) does not hold. The restriction here to a model basin of uniform depth is immaterial since coastally bound waves analogous to Kelvin waves occur when the bathymetry is variable and errors similar to those described can be expected in R–S models of real basins. The errors which may occur when modelling fully two-dimensional waves are practically beyond analysis, in view of the complicated form of (17).

7. A Modified Richardson–Sielecki Scheme

Relatively simple changes to the basic Richardson–Sielecki scheme (12) for Eqs. (1) can produce significant changes in the dispersion relation and hence in the fidelity with which waves phenomena can be modelled. Prandle [6] introduced a scheme, which when applied to Eqs. (1), takes the form

$$\frac{\eta_{mn}^{s+1} - \eta_{mn}^{s}}{\Delta t} = -h\left[\frac{u_{m+1,n}^{s+1/2} - u_{mn}^{s+1/2}}{\Delta x} + \frac{v_{m,n+1}^{s+1/2} - v_{mn}^{s+1/2}}{\Delta y}\right], \tag{26a}$$

$$\frac{u_{mn}^{s+3/2} - u_{mn}^{s+1/2}}{\Delta t} = -g\,\frac{\eta_{mn}^{s+1} - \eta_{m-1,n}^{s+1/2}}{\Delta x} + \frac{f}{4}\left[v_{m-1,n}^{s+3/2} + v_{m-1,n+1}^{s+3/2} + v_{mn}^{s+1/2} + v_{m,n+1}^{s+1/2}\right], \tag{26b}$$

$$\frac{v_{mn}^{s+3/2} - v_{mn}^{s+1/2}}{\Delta t} = -g\,\frac{\eta_{mn}^{s+1} - \eta_{m,n-1}^{s+1}}{\Delta y} - \frac{f}{4}\left[u_{mn}^{s+3/2} + u_{m,n-1}^{s+3/2} + u_{m+1,n}^{s+1/2} + u_{m+1,n-1}^{s+1/2}\right]. \tag{26c}$$

This scheme differs from (12) in that the velocities used in each Coriolis term now lie on a sloping plane in (x, y, t)-space. These averages are now time- and space-centred in the sense that the centroid of the four values used in each is coincident in space with the variable being updated and midway in time between the old and new time levels.

Use of (26) in place of (12) does not increase computing time and storage requirements, provided that the variables are evaluated in the following order at each time step:

(i) Elevations η_{nm}^{s+1} are first evaluated over the whole grid using Eq. (26a).

(ii) Columns of u and v values are computed alternately, using (26b) and (26c) in turn as the grid is swept in the positive x-direction, i.e., as m increases. For example, all $u_{mn}^{s+3/2}$ are computed before any $v_{mn}^{s+3/2}$; all $v_{mn}^{s+3/2}$ are computed before any $u_{m+1,n}^{s+3/2}$.

Reference to Fig. 2 will show that with this ordering, only the most recently computed value of each variable need be stored. The updated value of each variable can be stored in the location which held the previous value. On the understanding that the difference scheme (26) is used in conjunction with this left-to-right or "$+x$" sweep across the grid when computing velocities at each time step, it will be convenient to refer to (26) as the "$+x$ modified Richardson–Sielecki scheme" or even more briefly as the "$+x$ scheme."

8. Stability of the Modified Richardson–Sielecki Scheme

On substituting the fundamental solution (13) in Eqs. (26) and following the same procedure as in Section 5, it is found that stability of the $+x$ scheme depends on the roots of the quadratic

$$A\lambda^2 - B\lambda + A^* = 0, \tag{27}$$

where

$$\lambda = \exp(i\omega \cdot \Delta t),$$

$$A = 1 + \tfrac{1}{4}f^2 \Delta t^2 \cos^2 \tfrac{1}{2}l \cdot \Delta y \cdot \exp(-ik \cdot \Delta x),$$

$$B = 2 - \tfrac{1}{2}f^2 \Delta t^2 \cos^2 \tfrac{1}{2}l \cdot \Delta y - 4c^2 \Delta t^2 \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2}\right]$$

and A^* denotes the complex conjugate of A. For stability both roots of (27) must satisfy (15). If the roots are written as λ_1 and λ_2, it follows from (27) that

$$|\lambda_1| \cdot |\lambda_2| = |\lambda_1 \lambda_2| = |A^*/A| = 1 \tag{28}$$

and from this equation and (15) it follows that for stability, condition (16) must hold for both roots, which in turn implies (18), i.e., $|\sin \frac{1}{2}\omega \cdot \Delta t| \leqslant 1$. Equation (27) can be rearranged to give the following dispersion relation for the $+x$ scheme:

$$\sin^2 \tfrac{1}{2}\omega \cdot \Delta t = c^2 \Delta t^2 \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2} \right]$$
$$+ [\tfrac{1}{2} f \cdot \Delta t \cdot \cos \tfrac{1}{2}(\omega \cdot \Delta t - k \cdot \Delta x) \cdot \cos \tfrac{1}{2} l \cdot \Delta y]^2. \quad (29a)$$

It follows from (29a) by lengthy but routine analysis that the $+x$ modified R–S scheme (26) has the same stability limit (20) as the original R–S scheme (12).

9. Further Modified Richardson–Sielecki Schemes

The choice of direction in which to sweep across the grid when updating the velocities is obviously arbitrary. If the Coriolis terms in Eqs. (26b) and (26c) are replaced by

$$\frac{f}{4} [v_{m-1,n}^{s+1/2} + v_{m-1,n+1}^{s+1/2} + v_{mn}^{s+3/2} + v_{m,n+1}^{s+3/2}]$$

and

$$-\frac{f}{4} [u_{mn}^{s+1/2} + u_{m,n-1}^{s+1/2} + u_{m+1,n}^{s+3/2} + u_{m+1,n-1}^{s+3/2}]$$

a very similar scheme results, except that to maintain the same economy in storage it is necessary at each time step to sweep across the grid in the negative x-direction, calculating columns of v's and u's alternately for decreasing values of m. This will be called a "$-x$ modified R–S scheme" or "$-x$ scheme."

In fact, with the arrangement that updated values immediately overwrite previous values, no programming changes are involved in switching from a $+x$ scheme to a $-x$ scheme, other than changing the direction of sweep. For this reason it is quite practicable to program a model using the $+x$ and $-x$ schemes on alternate time steps so as to eliminate any bias caused by the directional nature of either scheme alone. Such an alternating sweep scheme will be termed a "$+x/-x$ modified R–S scheme" or "$+x/-x$ scheme."

With appropriate modifications to Eqs. (26), the grid could be swept in the positive or negative y-directions or alternately in both these directions, giving $+y$, $-y$ or $+y/-y$ modified R–S schemes. Indeed, since no programming change other than the order of computation is required, it would be possible to have any combination of the schemes so far mentioned.

Analysis of the $-x$ scheme shows that the dispersion relation is

$$\sin \tfrac{1}{2}\omega \cdot \Delta t = c^2 \Delta t^2 \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2} \right]$$

$$+ [\tfrac{1}{2}f \cdot \Delta t \cdot \cos \tfrac{1}{2}(\omega \cdot \Delta t + k \cdot \Delta x) \cos \tfrac{1}{2}l \cdot \Delta y]^2. \qquad (29b)$$

The maximum permissible time step can be derived from this equation and is given by (20); i.e., the stability limit is the same as that for the $+x$ scheme. It follows that the same criterion applies to the $+x/-x$ scheme, which consists of alternate application of $+x$ and $-x$ schemes. In fact, it can be shown similarly that (20) holds for $+y$, $-y$ and $+y/-y$ schemes also.

10. Kelvin Wave Simulation with Modified Richardson–Sielecki Schemes

Again, as a first test of the ability of modified R–S schemes to represent solutions of the shallow-water equations (1), consider the discrete analogs of the Kelvin waves of Section 2, that is, coastally bound travelling waves with zero transverse velocity and amplitude decaying exponentially in the seaward direction.

A wave progressing in the positive x-direction in a model using the $+x$ modified R–S scheme obeys the dispersion relation (29b). For relatively long waves, in the sense discussed in Section 6, the dispersion relations (29a) and (29b) reduce to the dispersion relation (3) of the continuous case. Thus the modified R–S schemes show an improvement over the original R–S scheme, which led to the faulty dispersion relation (21).

Since the dispersion relations for $+x$ and $-x$ schemes both reduce to (3) in the relatively long wave approximation, it is necessary to consider (29a) and (29b) in full in order to discover whether there is any directional bias in $+x$ and $-x$ schemes and hence whether there is any advantage is using the $+x/-x$ scheme. It can easily be verified by substitution in (26) that a wave travelling in either x-direction with $v = 0$ satisfies

$$\sin^2 \tfrac{1}{2}\omega \cdot \Delta t = c^2 \Delta t^2 \frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2}. \qquad (30)$$

Thus in order to obey the appropriate dispersion relation (29a), such a wave progressing in the positive $x$direction in a $+x$ scheme must also satisfy

$$c^2 \Delta t^2 \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2} + [\tfrac{1}{2}f \cdot \Delta t \cdot \cos \tfrac{1}{2}(\omega \cdot \Delta t + |k| \cdot \Delta x) \cos \tfrac{1}{2}l \cdot \Delta y]^2 = 0. \qquad (31)$$

This leads to the following expression for l in terms of k and ω:

$$l = i\frac{2}{\Delta y}\tanh^{-1}\left[\frac{f\cdot\Delta y}{2c}\cos\tfrac{1}{2}(\omega\cdot\Delta t + |k|\,\Delta x)\right]. \qquad (32)^2$$

On the other hand, a wave progressing in the positive x-direction in a $-x$ scheme satisfies (29a) and (30) and hence

$$l = i\frac{2}{\Delta y}\tanh^{-1}\left[\frac{f\cdot\Delta y}{2c}\cos\tfrac{1}{2}(\omega\cdot\Delta t - |k|\,\Delta x)\right]. \qquad (33)^2$$

As a numerical illustration, consider again the example discussed in Section 6. Equation (30) yields $k = 5.1839 \times 10^{-6}\ \mathrm{m}^{-1}$ and with this value, Eq. (32) gives

$$l = i4.4665 \times 10^{-6}\ \mathrm{m}^{-1} \qquad (34)$$

while Eq. (33) gives

$$l = i4.5161 \times 10^{-6}\ \mathrm{m}^{-1}. \qquad (35)$$

For the same values of f and c, the value of l in the corresponding continuous case is (from (4)):

$$l = i4.5088 \times 10^{-6}\ \mathrm{m}^{-1}. \qquad (36)$$

Since the results from the $+x$ and $-x$ schemes, (34) and (35), bracket the desired result (36), it is to be expected that a similar wave in the $+x/-x$ scheme with the same model parameters should have a value of l close to (36).

In order to determine l for the $+x/-x$ scheme, the following numerical experiment was performed using a model of a channel parallel to the x-axis, of width $16\cdot\Delta y$ and length $300\cdot\Delta x$. The latter dimension was chosen to ensure that the computations in a central section of the channel, approximately one wavelength $(36\cdot\Delta x)$ long, were unaffected by the boundary conditions at the ends of the channel for a least two complete cycles (approximately 102 time steps). A value of l intermediate between (35) and (36) was assumed and initial conditions were calculated from the formulas

$$\eta^{s}_{mn} = e^{iln\cdot\Delta y}\sin[\omega s\cdot\Delta t - |k|\,m\cdot\Delta x], \qquad (37a)$$

$$u^{s+1/2}_{mn} = (g/c)e^{iln\cdot\Delta y}\sin[\omega(s+\tfrac{1}{2})\Delta t - |k|\,(m-\tfrac{1}{2})\Delta x], \qquad (37b)$$

$$v^{s+1/2}_{mn} = 0, \qquad (37c)$$

which represents a progressive Kelvin wave with elevation of unit magnitude. The model was run for two cycles during which the sum of the squares of the differences

[2] For numerical evaluation of (32) and (33) it is convenient to use $\tanh^{-1}x = \tfrac{1}{2}\ln|(1+x)/(1-x)|$ if $x < 1$.

between the model elevations and those defined by (37a) were summed over all grid-points in the central section (one wavelength) of the model. The value of l was adjusted during subsequent runs until this accumulated square error was minimized, which occurred when

$$l = i4.491 \times 10^{-6}\,\mathrm{m}^{-1}. \tag{38}$$

In this case then, the value of l for the $+x/-x$ scheme is approximately equal to the average of the values of l, (34) and (35), for the $+x$ and $-x$ schemes. With this value of l, the maximum discrepancy between elevations computed with the model and those given by (37a) remained less than 0.005 m in the course of two complete cycles. During the same period the magnitude of transverse velocity v differed from zero by less than 0.002 m/sec. (For comparison, the peak value of longitudinal velocity u was 0.362 m/sec.)

The value (38) is intermediate between (34) and (35) and satisfactorily close to the continuous case (36). No general conclusion can be drawn from this single experiment but the results encourage the expectation that the $+x/-x$ scheme is preferable to the $+x$ or $-x$ schemes for general purposes.

In an actual model, coastlines may be at any orientation to the coordinate axes. At the opposite extreme to the cases just discussed, one can take a Kelvin-type wave travelling along a coastline parallel to the y-axis in a model with $+x$, $-x$ or $+x/-x$ schemes, that is, the direction of travel of the wave is perpendicular to the direction in which the grid is swept in the difference calculations rather than parallel, as above. Equivalently, one can consider the same physical configuration as before, a wave travelling along a coast parallel to the x-axis but with $+y$, $-y$ or $+y/-y$ modified R–S schemes. The general dispersion relations are

$$\sin^2 \tfrac{1}{2}\omega \cdot \Delta t = c^2 \Delta t \left[\frac{\sin^2 \frac{1}{2}k \cdot \Delta x}{\Delta x^2} + \frac{\sin^2 \frac{1}{2}l \cdot \Delta y}{\Delta y^2} \right] + \left[\tfrac{1}{2}f \cdot \Delta t \cdot \cos \tfrac{1}{2}k \cdot \Delta x \cos \tfrac{1}{2}(\omega \cdot \Delta t \mp l \cdot \Delta y)\right]^2 \tag{39}$$

with $+y$ and $-y$ schemes, respectively. For Kelvin type waves travelling parallel to the x-axis, Eq. (30) again holds and can be subtracted from (39). After some further algebra, one finds that for Kelvin waves travelling in the positive x-direction,

$$l = \frac{2}{\Delta y} \tan^{-1} \left[-\cos \tfrac{1}{2}\omega \cdot \Delta t \left\{ \pm \sin \tfrac{1}{2}\omega \cdot \Delta t + i \frac{2c}{f \cdot \Delta y \cdot \cos \frac{1}{2} |k| \Delta x} \right\}^{-1} \right]. \tag{40}$$ [3]

For example, with the numerical values quoted earlier in this section, Eq. (40) gives

$$l = \mp 2.1392 \times 10^{-8} + i4.4912 \times 10^{-6}\,\mathrm{m}^{-1}, \tag{41}$$

[3] Numerical evaluation of l from (40) is facilitated by using

$$\tan^{-1}(a + ib) = n\pi + \frac{1}{2}\left[\tan^{-1}\frac{a}{1-b} + \tan^{-1}\frac{a}{1+b}\right] - \frac{i}{2}\ln\left[\frac{(1-b)^2 + a^2}{(1+b)^2 + a^2}\right]^{1/2}.$$

where the negative and positive signs apply to $+y$ and $-y$ schemes, respectively. Since the real part of l is non-zero, waves are slewed (see Fig. 3) but the effect is an order of magnitude smaller than with the original R–S scheme (see Section 6). Also, $+y$ and $-y$ schemes produce slewing in opposite directions, which suggests that in the alternating $+y/-y$ scheme, these undesirable effects may cancel out. This was confirmed in numerical tests on a model similar to that used in the previous tests. With initial conditions computed using

$$l = i4.4912 \times 10^{-6}\,\mathrm{m}^{-1}$$

in Eqs. (37), an unslewed Kelvin wave was produced which propagated without significant change of form for the two-cycle duration of the test with the $+y/-y$ scheme. During this period, errors in u and v were slightly smaller than in the corresponding test with the $+x/-x$ scheme.

11. Conclusions

It has been shown that the plain Richardson–Sielecki (R–S) finite-difference scheme for the shallow-water equations can distort simulated long waves and that relatively simple modification of the scheme apparently eliminates this undesirable feature. The effects in question are discussed purely in the context of Kelvin waves in a linearized model with uniform mean water depth and straight coastlines. However, as the gravity and Coriolis terms were retained in the equations used and Kelvin waves (or analogous coastally bound waves) play a major role in most shallow-water models, it is reasonable to expect that similar distortion occurs when the R–S scheme is used in models with realistic bottom topography, friction and irregular coastlines. Possible distortion of fully two-dimensional long waves by the R–S scheme has not yet been investigated.

The basic feature of the modified R–S scheme discussed in this paper consists of rearranging the order of computation so that the velocity components are computed in turn column by column as the model grid is swept in the positive or negative x-directions or row by row in the positive or negative y-directions. It is shown that some improvement in Kelvin wave simulation is achieved by executing the computational sweep in opposite directions on alternate time steps. It seems reasonable to suppose that removing directional bias on the computation by this means will improve the accuracy of more general realistic models also.

The original R–S scheme achieves considerable storage and computing economies by requiring only the most recently computed values of any variable at any given stage of the computation. The modified R–S schemes preserve this useful property, even when a version with alternating directions of sweep is used.

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